

Solutions to Maths workbook - 1 | Quadratic Equations

Level – 3

Daily Tutorial Sheet – 13

155. Here $3x^3 = \{x^2 + (\sqrt{18}x + \sqrt{32})\} \{x^2 - (\sqrt{18}x + \sqrt{32})\} - 4x^2 = x^4 - (\sqrt{18}x + \sqrt{32})^2 - 4x^2$
or $3x^3 + 4x^2 = x^4 - 2(3x + 4)^2$ or $x^4 - 2(3x + 4)^2 - x^2 - x^2(3x + 4) = 0$
Put $3x + 4 = y$; then the equation becomes $x^4 - yx^2 - 2y^2 = 0$
or $x^4 - 2yx^2 + yx^2 - 2y^2 = 0$ or $x^2(x^2 - 2y) + y(x^2 - 2y) = 0$
or $(x^2 - 2y)(x^2 + y) = 0$ $\therefore x^2 - 2y = 0$ or $x^2 + y = 0$
If $x^2 - 2y = 0$ then $x^2 - 2(3x + 4) = 0$
or $x^2 - 6x - 8 = 0$ $\therefore x = \frac{6 \pm \sqrt{36 + 32}}{2} = \frac{6 \pm \sqrt{68}}{2} = 3 \pm \sqrt{17}$.
If $x^2 + y = 0$ then $x^2 + 3x + 4 = 0$ $\therefore x = \frac{-3 \pm \sqrt{9 - 16}}{2} = \frac{-3 \pm \sqrt{7}i}{2}$
Thus $x = 3 \pm \sqrt{17}, \frac{-3 \pm \sqrt{7}i}{2}$.

156. Let $f(x) = \frac{a}{2}x^2 + bx + c$
 α is a root of $ax^2 + bx + c = 0$
 $\Rightarrow a\alpha^2 + b\alpha + c = 0$
 $\Rightarrow \frac{a}{2}\alpha^2 + b\alpha + c = -\frac{a}{2}\alpha^2$... (i)
 β is a root of $-ax^2 + bx + c = 0$
 $\Rightarrow -a\beta^2 + b\beta + c = 0 \Rightarrow \frac{a}{2}\beta^2 + b\beta + c = \frac{3a}{2}\beta^2$ (2)
If $a > 0$, $\frac{a}{2}\alpha^2 + b\alpha + c < 0$ and $\frac{a}{2}\beta^2 + b\beta + c > 0$ {using (1) and (2)}
 $\therefore f(\alpha) < 0$ and $f(\beta) > 0$.
If $a < 0$, $\frac{a}{2}\alpha^2 + b\alpha + c > 0$ and $\frac{a}{2}\beta^2 + b\beta + c < 0$
 $\therefore f(\alpha) > 0$ and $f(\beta) < 0$
 $\therefore f(\alpha)$ and $f(\beta)$ are of opposite signs
 $\therefore f(\gamma) = 0$ where γ lies between α and β or γ is a root of $\frac{a}{2}x^2 + bx + c = 0$
Where γ lies between α and β .

157. Let $\frac{(x-1)\cos^2 \alpha}{x^2 - \cos^2 \alpha} = y$ or $(x-1)\cos^2 \alpha = yx^2 - y\cos^2 \alpha$ or $yx^2 - \cos^2 \alpha \cdot x + \cos^2 \alpha \cdot (1-y) = 0$
As x is real, $D \geq 0$
 $\Rightarrow \cos^4 \alpha - 4y(1-y)\cos^2 \alpha \geq 0$ or $4y^2 - 4y + \cos^2 \alpha \geq 0$ ($\because \cos^2 \alpha > 0$).
Roots of the corresponding equation are $y = \frac{4 \pm \sqrt{16 - 16\cos^2 \alpha}}{8} = \frac{1 \pm \sqrt{1 - \cos^2 \alpha}}{2} = \frac{1 \pm \sin \alpha}{2}$
 $\therefore$ by the sign-scheme we know that y must not lie between the roots, i.e.,

$$\frac{1}{2}(1 - \sin \alpha) \text{ and } \frac{1}{2}(1 + \sin \alpha), \text{ i.e., } \frac{1}{2}\left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}\right)^2 \text{ and } \frac{1}{2}\left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\right)^2.$$

158. As every pair has a common root, let

$$x^2 - b_1x + c_1 = 0 \text{ have the roots } \alpha, \beta$$

$$x^2 - b_2x + c_2 = 0 \text{ have the roots } \alpha, \gamma$$

$$x^2 - b_3x + c_3 = 0 \text{ have the roots } \beta, \gamma$$

$$\therefore \alpha + \beta = b_1 \quad \dots\dots(1)$$

$$\alpha\beta = c_1 \quad \dots\dots(2)$$

$$\alpha + \gamma = b_2 \quad \dots\dots(3)$$

$$\alpha\gamma = c_2 \quad \dots\dots(4)$$

$$\beta + \gamma = b_3 \quad \dots\dots(5)$$

$$\beta\gamma = c_3 \quad \dots\dots(6)$$

$$\text{Now, LHS} = b_1^2 + b_2^2 + b_3^2 + 4(c_1 + c_2 + c_3) = (\alpha + \beta)^2 + (\alpha + \gamma)^2 + (\beta + \gamma)^2 + 4(\alpha\beta + \alpha\gamma + \beta\gamma) \quad \{\text{using (1) through (6)}\}$$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) + 6(\alpha\beta + \alpha\gamma + \beta\gamma) = 2(\alpha^2 + \beta^2 + \gamma^2 + 3(\alpha\beta + \beta\gamma + \gamma\alpha))$$

$$\text{RHS} = 2(b_1b_2 + b_2b_3 + b_3b_1) = 2\{(\alpha + \beta)(\alpha + \gamma) + (\alpha + \gamma)(\beta + \gamma) + (\beta + \gamma)(\alpha + \beta)\}$$

$$2\{(\alpha^2 + \alpha\gamma + \beta\alpha + \beta\gamma) + (\gamma^2 + \alpha\beta + \alpha\gamma + \gamma\beta) + (\beta^2 + \beta\alpha + \gamma\alpha + \gamma\beta)\} = 2(\alpha^2 + \beta^2 + \gamma^2 + 3\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\therefore \text{LHS} = \text{RHS}$$

159. Here $1 \geq \log_5(ax^2 + 4x + a) - \log_5(x^2 + 1)$ or $1 \geq \log_5 \frac{ax^2 + 4x + a}{x^2 + 1}$ or $\log_5 5 \geq \log_5 \frac{ax^2 + 4x + a}{x^2 + 1}$

$$\therefore \text{we have } 5 \geq \frac{ax^2 + 4x + a}{x^2 + 1} \quad \dots\dots(1)$$

Also, logarithms to be real, $x^2 + 1 > 0$ and $ax^2 + 4x + a > 0$

$$x^2 + 1 > 0 \text{ is true for all } x \in R.$$

$$\text{Now, } ax^2 + 4x + a > 0 \quad \dots\dots(2)$$

$\therefore$ the values of the parameter $a \in R$ should be such that both (1) and (2) hold for all $x \in R$.

Now, (2) holds for all $x \in R$ if $D = 4^2 - 4.a.a < 0$ and coefficient of $x^2 = a > 0$,

$$\text{i.e., } 16 - 4a^2 < 0, a > 2. \Rightarrow a > 2$$

$$\text{Now, from (1), } 5(x^2 + 1) \geq ax^2 + 4x + a \quad (\because x^2 + 1 > 0)$$

$$\text{or } (5 - a)x^2 - 4x + 5 - a \geq 0.$$

$$\text{This holds for all } x \in R \text{ if } D = (-4)^2 - 4(5 - a)(5 - a) \leq 0$$

$$\text{and the coefficient of } x^2 = 5 - a > 0, \text{ i.e., } 16 - 4(5 - a)^2 \leq 0, 5 - a > 0$$

$$\Rightarrow (5 - a)^2 \geq 4, a < 5 \Rightarrow (5 - a \leq -2 \text{ or } 5 - a \geq 2), a < 5 \Rightarrow \{a \geq 7 \text{ or } a \leq 3\}, a < 5 \Rightarrow a \leq 3$$

Hence, the required values of a satisfy $a > 2$ and $a \leq 3$

$$\therefore \text{the solution-set} = (2, 3]$$

160. Here $4^x + 3 - a(2^x + 1) \leq 0 \quad \dots\dots(1)$

As $4^x + 3$ & $2^x + 1$ are positive.

Hence (1) only when $a > 0$

$$\text{As } 4^x - a.2^x - a + 3$$

$$= (2^x)^2 - 2 \cdot \frac{a}{2} \cdot 2^x + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 - a + 3$$

$$= \left(2^x - \frac{a}{2}\right)^2 - \left(\frac{a^2}{4} + a - 3\right) \leq 0, \quad (\text{given})$$

$$\therefore \frac{a^2}{4} + a - 3 \geq \left(2^x - \frac{a}{2}\right)^2 \geq 0 \quad \text{or} \quad a^2 + 4a - 12 \geq 0$$

Corresponding equation is $a^2 + 4a - 12 = 0$

or $(a + 6)(a - 2) = 0 ; \quad \therefore \quad a = -6, 2$

$\therefore$ sign-scheme for $a^2 + 4a - 12, a \in R$ is as follows:

$$\begin{array}{ccccccc} & (+) & & (-) & & & (+) \\ & | & & | & & & | \\ \hline & -6 & & & & & 2 \end{array}$$

$\therefore \quad a^2 + 4a - 12 \geq 0$ when $a \leq -6$ or $a \geq 2$

But $a > 0$; so $a \geq 2$